Spectral Gap and Decay of Correlations in U(1)-Symmetric Lattice Systems in Dimensions $D < 2$

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We consider many-body systems with a global U(1) symmetry on a class of lattices with the (fractal) dimensions $D < 2$ and their zero temperature correlations whose observables behave as a vector under the U(1) rotation. For a wide class of the models, we prove that if there exists a spectral gap above the ground state, then the correlation functions have a stretched exponentially decaying upper bound. This is an extension of the McBryan-Spencer method at finite temperatures to zero temperature. The class includes quantum spin and electron models on the lattices, and our method also allows finite or infinite (quasi)degeneracy of the ground state. The resulting bounds rule out the possibility of the corresponding magnetic and electric long-range order.

1 Introduction

As is well known, low dimensional systems show large fluctuations for continuous symmetry. The most famous result is the Hohenberg-Mermin-Wagner theorem [1] which states that the corresponding spontaneous magnetizations are vanishing at finite temperatures in one and two dimensions. Since their articles appeared, their method have been applied to various systems [2] including classical and quantum magnets, interacting electrons in a metal and Bose gas.[1] The theorem was extended to the models on a class of generic lattices with the fractal dimensions $D \le 2$ by Cassi [4]. In a stronger sense, it was also proved for a class of low-dimensional systems that the equilibrium states are invariant under the action of the continuous symmetry group [5, 6]. Even at zero temperature, the same is true [7, 8, 9] if the corresponding one- or two-dimensional system satisfies conditions [10] such as boundedness of susceptibilities.[2]

About the corresponding long range correlations, Fisher and Jasnow [11] proved clustering properties of two point functions by using the Bogoliubov inequality. See also ref. [6]. McBryan and Spencer [12] obtained a better decay for two point correlations of classical spin systems. Their method has been applied various classical and quantum systems. The resulting upper bounds for the correlations decay by power, exponential or stretched exponential laws [13, 14, 15, 16, 17, 18] and rule out the ordering at finite temperatures in (fractal) dimensions $D \le 2$. But a zero temperature analogue of the McBryan-Spencer bound has not yet been obtained.

On the other hand, since Haldane [19] predicted a “massive” phase in low dimensional, isotropic quantum systems, many examples have been found to have a spectral gap above the ground state and exponentially decaying correlations in the ground state as initiated by [20]. See also related articles [21, 22, 23]. These examples raise the following question: Consider a generic low dimensional system with a continuous symmetry and its zero temperature correlations whose observables show a non-trivial representation under the action of the continuous symmetry group. Then a spectral gap above the ground state implies (stretched) exponential decay of the correlations? And if so, the corresponding upper bound can be obtained by the McBryan-Spencer method?

In this paper, we address this problem. We consider the correlation functions whose observables behave as a vector under a U(1) rotation. In order to estimate the decay of the correlation functions, we extend the McBryan-Spencer method to zero temperature under the assumption that there exists a spectral gap above the ground state. As a result, we prove that the correlation functions have a stretched exponentially decaying upper bound. This method covers a wide class of many-body systems with a global U(1) symmetry on a class of lattices with the (fractal) dimensions $D < 2$. We stress that this method is also an extension of the Combes-Thomas method [24] for Schrödinger operators to many-body systems in statistical mechanics. In the next section, we will give the precise definition of the class of the fractal lattices and describe the results for two typical examples, the

[1] For a mathematically rigorous treatment for the unbounded operators, see ref. [3].

[2] Since a single spin shows the spontaneous magnetization at zero temperature, the absence of the spontaneous symmetry breaking implies that the strong fluctuations due to the interaction destroy the ordering and lead to the finite susceptibilities. In other words, one cannot expect the absence of spontaneous symmetry breaking at zero temperature in a generic situation.

Heisenberg and the Hubbard models. The proof of the main results will be given in Section 3.

2 Models and results

We begin with defining the class of the (fractal) lattices which we consider in this paper. The class of the lattices is the same as in [18]. See also [25] for fractal lattices and models on the lattices.

Consider first a connected lattice $\Lambda = (\Lambda_s, \Lambda_b)$, where Λ_s is a set of sites, $i, j, k, \ell, \ldots$, and Λ_b is a set of bonds, i.e., pairs of sites, $\{i, j\}, \{k, \ell\}, \ldots$. If a sequence of sites, $i_0, i_1, i_2, \ldots, i_r$, satisfies $\{i_{n-1}, i_n\} \in \Lambda_b$ for $n = 1, 2, \ldots, r$, then we say that the path, $\{i_0, i_1, i_2, \ldots, i_r\}$, has length r and connects i_0 to i_r. We define the "sphere", $S_r(m)$, centered at $m \in \Lambda_s$ with the radius r as

$$S_r(m) := \{\ell \in \Lambda_s | \mathrm{dist}(\ell, m) = r\}, \tag{2.1}$$

where $\mathrm{dist}(\ell, m)$ is the graph-theoretic distance which is defined to be the shortest path length that one needs to connect ℓ to m. Let $|A|$ denote the number of the elements in the set A. We assume that there exists a "(fractal) dimension" $D \geq 1$ of the lattice Λ such that the number $|S_r(m)|$ of the sites in the sphere satisfies

$$\sup_{m \in \Lambda_s} |S_r(m)| \leq C_0 r^{D-1} \tag{2.2}$$

with some positive constant C_0.

We consider spin or fermion systems with a global U(1) symmetry on the lattice Λ with the (fractal) dimensions $1 \leq D < 2$. We require the existence of a "uniform gap" above the sector of the ground state of the Hamiltonian H_Λ. The precise definition of the "uniform gap" is:

Definition 2.1 *We say that there is a uniform gap above the sector of the ground state if the spectrum $\sigma(H_\Lambda)$ of the Hamiltonian H_Λ satisfies the following conditions: The ground state of the Hamiltonian H_Λ is q-fold (quasi)degenerate in the sense that there are q eigenvalues, $E_{0,1}, \ldots, E_{0,q}$, in the sector of the ground state at the bottom of the spectrum of H_Λ such that*

$$\Delta\mathcal{E} := \max_{\mu,\mu'} \{|E_{0,\mu} - E_{0,\mu'}|\} \to 0 \quad \text{as} \quad |\Lambda_s| \to \infty. \tag{2.3}$$

Further the distance between the spectrum, $\{E_{0,1}, \ldots, E_{0,q}\}$, of the ground state and the rest of the spectrum is larger than a positive constant ΔE which is independent of the volume $|\Lambda_s|$. Namely there is a spectral gap ΔE above the sector of the ground state.

Remark: For the special case with $q = 1$, the ground state is unique. As well known examples with $q \neq 1$, Majumdar-Ghosh model [21] shows a spectral gap above the degenerate ground state, and the spin-1 antiferromagnetic chain with open boundaries exhibits a spectral gap above the fourfold quasidegenerate ground state [22]. In the thermodynamic limit $|\Lambda_s| \to \infty$, we allow infinite degeneracy $q = \infty$ of the ground state as in [26].

Consider first a quantum spin system with a U(1) symmetry on the lattice Λ. As a concrete example, we consider the standard XXZ Heisenberg model on the lattice. The Hamiltonian H_Λ is given by

$$H_\Lambda = H_\Lambda^{XY} + V_\Lambda(\{S_i^z\}) \tag{2.4}$$

with

$$H_\Lambda^{XY} = 2 \sum_{\{i,j\}\in\Lambda_b} J_{i,j}^{\rm XY}(S_i^x S_j^x + S_i^y S_j^y), \tag{2.5}$$

where (S_i^x, S_i^y, S_i^z) is the spin operator at the site $i \in \Lambda_s$ with the spin $S = 1/2, 1, 3/2, \ldots$, and $J_{i,j}^{\rm XY}$ are real coupling constants; $V_\Lambda(\{S_i^z\})$ is a real function of the z-components, $\{S_i^z\}_{i\in\Lambda_s}$, of the spins. For simplicity, we take

$$V_\Lambda(\{S_i^z\}) = \sum_{\{i,j\}\in\Lambda_b} J_{i,j}^{\rm Z} S_i^z S_j^z \tag{2.6}$$

with real coupling constants $J_{i,j}^{\rm Z}$. We assume that there are positive constants, $J_{\max}^{\rm XY}$ and $J_{\max}^{\rm Z}$, which satisfy $|J_{i,j}^{\rm XY}| \le J_{\max}^{\rm XY}$ and $|J_{i,j}^{\rm Z}| \le J_{\max}^{\rm Z}$ for any bond $\{i,j\} \in \Lambda_b$. We stress that we can also treat more general interactions in the same way.

Let P_0 be the projection onto the sector of the ground state. We define the ground-state expectation as

$$\langle \cdots \rangle_0 := \frac{1}{q} {\rm Tr}\ (\cdots) P_0, \tag{2.7}$$

where Tr stands for the trace which is over all the spin states. We consider the transverse spin-spin correlation, $\langle S_m^+ S_n^- \rangle_0$, where $S_i^\pm := S_i^x \pm i S_i^y$.

Theorem 2.2 *Suppose that $1 \le D < 2$ and that there is a uniform gap ΔE above the sector of the ground state in the sense of Definition 2.1. Then there exists a positive constant γ such that the transverse spin-spin correlation satisfies the bound,*

$$\left| \left\langle S_m^+ S_n^- \right\rangle_0 \right| \le {\rm Const.} \exp\left[-\gamma \{{\rm dist}(m,n)\}^{1-D/2} \right], \tag{2.8}$$

in the thermodynamic limit $|\Lambda_s| \to \infty$.

Remark: 1. It is easy to extend the result to more complicated correlations such as the multispin correlation $\left\langle S_{m_1}^+ \cdots S_{m_j}^+ S_{n_1}^- \cdots S_{n_j}^- \right\rangle_0$.
2. Recently, the exponential clustering of the correlations was proved for quantum many-body lattice systems by Hastings [27, 28] under the gap assumption. This is a non-relativistic version of Fredenhagen's theorem [29] of relativistic quantum field theory. Combining this exponential clustering with the present result, a better, exponentially decaying bound for the ground-state correlations can be obtained [30] for a class of models on lattices with the (fractal) dimensions $D < 2$ and with a certain self-similarity. The class includes the translationally invariant regular lattices such as $\mathbf{Z}$.

For a finite volume $|\Lambda_s| < \infty$, the Hamiltonian H_Λ of (2.4) commutes with $S_\Lambda^z := \sum_{i\in\Lambda_s} S_i^z$. Using this symmetry, we can block diagonalize the Hamiltonian H_Λ, and denote by $H_{\Lambda,M}$ the restriction of H_Λ to the eigenspace $\mathcal{H}_{\Lambda,M}$ of S_Λ^z with the eigenvalue M. Let $P_{0,M}$ denote the projection onto the ground state of $H_{\Lambda,M}$ in the subspace $\mathcal{H}_{\Lambda,M}$. Here

the ground state may be (quasi)degenerate in the sense of Definition 2.1. We define the ground-state expectation as

$$\langle \cdots \rangle_{0,M} := \frac{1}{q_M} \mathrm{Tr}\ (\cdots) P_{0,M}, \tag{2.9}$$

where q_M is the degeneracy of the sector of the ground state of $H_{\Lambda,M}$.

Theorem 2.3 *Suppose that $1 \le D < 2$ and that there is a uniform gap ΔE above the sector of the ground state of $H_{\Lambda,M}$ in the spectrum of $H_{\Lambda,M}$ with the eigenvalue M of S_Λ^z in the sense of Definition 2.1. Then there exists a positive constant γ such that the transverse spin-spin correlation satisfies the bound,*

$$\left| \left\langle S_m^+ S_n^- \right\rangle_{0,M} \right| \le \mathrm{Const.} \exp\left[-\gamma \{ \mathrm{dist}(m,n) \}^{1-D/2} \right], \tag{2.10}$$

in the thermodynamic limit $|\Lambda_s| \to \infty$.

As an example of a lattice fermion system, we consider the following Hamiltonian [16, 17] on the lattice Λ with the (fractal) dimension $1 \le D < 2$:

$$H_\Lambda = - \sum_{\{i,j\} \in \Lambda_b} \sum_{\mu = \uparrow, \downarrow} \left(t_{i,j} c_{i,\mu}^\dagger c_{j,\mu} + t_{i,j}^* c_{j,\mu}^\dagger c_{i,\mu} \right) + V(\{n_{i,\mu}\}) + \sum_{i \in \Lambda_s} \mathbf{B}_i \cdot \mathbf{S}_i, \tag{2.11}$$

where $c_{i,\mu}^\dagger, c_{i,\mu}$ are, respectively, the electron creation and annihilation operators with the z component of the spin $\mu = \uparrow, \downarrow$, $n_{i,\mu} = c_{i,\mu}^\dagger c_{i,\mu}$ is the corresponding number operator, and $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ are the spin operator given by $S_i^a = \sum_{\mu,\nu=\uparrow,\downarrow} c_{i,\mu}^\dagger \sigma_{\mu,\nu}^a c_{i,\nu}$ with the Pauli spin matrix $(\sigma_{\mu,\nu}^a)$ for $a = x, y, z$; $t_{i,j} \in \mathbf{C}$ are the hopping amplitude, $V(\{n_{i,\mu}\})$ is a real function of the number operators, and $\mathbf{B}_i = (B_i^x, B_i^y, B_i^z) \in \mathbf{R}^3$ are local magnetic fields. We assume that the interaction $V(\{n_{i,\mu}\})$ is of finite range in the sense of the graph theoretic distance.

Clearly the Hamiltonian H_Λ of (2.11) commutes with the total number operator $\mathcal{N}_\Lambda = \sum_{i \in \Lambda_s} \sum_{\mu = \uparrow,\downarrow} n_{i,\mu}$ for a finite volume $|\Lambda_s| < \infty$. We denote by $H_{\Lambda,N}$ the restriction of H_Λ onto the eigenspace of $\mathcal{N}_\Lambda$ with the eigenvalue N. Let $P_{0,N}$ be the projection onto the sector of the ground state of $H_{\Lambda,N}$, and we denote the ground-state expectation by

$$\langle \cdots \rangle_{0,N} = \frac{1}{q_N} \mathrm{Tr}\ (\cdots) P_{0,N}, \tag{2.12}$$

where q_N is the degeneracy of the ground state. Assume the existence of a uniform gap above the ground state. Then we have

$$\left| \left\langle c_{m,\mu}^\dagger c_{n,\mu} \right\rangle_{0,N} \right| \le \mathrm{Const.} \exp\left[-\gamma \{ \mathrm{dist}(m,n) \}^{1-D/2} \right] \tag{2.13}$$

and

$$\left| \left\langle c_{m,\uparrow}^\dagger c_{m,\downarrow}^\dagger c_{n,\uparrow} c_{n,\downarrow} \right\rangle_{0,N} \right| \le \mathrm{Const.} \exp\left[-2\gamma \{ \mathrm{dist}(m,n) \}^{1-D/2} \right] \tag{2.14}$$

with some constant γ in the thermodynamic limit $|\Lambda_s| \to \infty$. If the local magnetic field has the form $\mathbf{B}_i = (0, 0, B_i)$, then we further have

$$\left| \left\langle S_m^+ S_n^- \right\rangle_{0,N} \right| \le \mathrm{Const.} \exp\left[-\gamma' \{ \mathrm{dist}(m,n) \}^{1-D/2} \right] \tag{2.15}$$

with some constant γ'. We remark that, in the latter situation, we can also restrict the Hamiltonian $H_{\Lambda,N}$ and the expectation to the subspace with a fixed total magnetization.

3 Proof of the main theorems

We will give a proof only for Theorem 2.3 because the rest of the bounds can be proved in the same way. The application to other systems is also straightforward.

Before proceeding to the proof, let us sketch the idea of the proof and the key tools. In the previous work [16], the global quantum mechanical U(1) symmetry was used for estimating the correlation functions of the Hubbard model at finite temperatures. Roughly speaking, the strength of the long-range ordering can be measured by twisting the U(1) phase locally in the pure imaginary direction. The difference between zero temperature and finite temperatures is in their density matrices. Since the projection $P_{0,M}$ onto the sector of the ground state can be written in the contour integral of the resolvent, we apply the method of [16] to the resolvent $(z - H_\Lambda)^{-1}$ instead of the Boltzmann weight for finite temperatures. It is well known that, for a single-particle resolvent with a classically forbidden energy, the Combes-Thomas method [24] yields the WKB-type tunnelling estimate, i.e., the exponentially decaying bound. In the method, twisting the quantum mechanical phase locally in the pure imaginary direction plays an important role, too, and so the Combes-Thomas method is essentially equivalent to the McBryan-Spencer method. In the present paper, we use the improved version [31] of the Combes-Thomas method. In order to extend the Combes-Thomas method to many-body systems in statistical mechanics, we further employ the technique [32, 33] which was developed for many-body systems to treat quantities of order of the volume.

In order to estimate the transverse spin-spin correlation, $\langle S_m^+ S_n^- \rangle_{0,M}$, we introduce "gauge transformation",

$$G(\alpha) := \prod_{i \in \Lambda_s} \exp[\alpha \theta_i S_i^z], \tag{3.1}$$

where α is a real parameter to be determined. For the choice of the function θ_i, we follow Picco's idea [14], but we modify it in order to obtain a better decay bound for the correlation. Let κ be a positive parameter satisfying

$$1 - \frac{D}{2} < \kappa < \frac{3}{2} - \frac{D}{2}, \tag{3.2}$$

and write $R = \mathrm{dist}(n, m)$. We choose the real function θ_i on the lattice Λ_s as

$$\theta_\ell = R^{1-D/2} \times \begin{cases} R^{-\kappa} - 1, & \text{for } \ell = m; \\ [\mathrm{dist}(\ell, m)/R]^\kappa - 1, & \text{for } 1 \le \mathrm{dist}(\ell, m) \le R; \\ 0, & \text{for } \mathrm{dist}(\ell, m) > R. \end{cases} \tag{3.3}$$

Note that

$$G(\alpha)^{-1} S_i^\pm G(\alpha) = e^{\pm \alpha \theta_i} S_i^\pm \quad \text{for} \quad i \in \Lambda_s. \tag{3.4}$$

Using the relation (3.4), we have

$$\begin{aligned} \mathrm{Tr}\ S_m^+ S_n^- P_{0,M} &= \mathrm{Tr}\ G(\alpha)^{-1} S_m^+ G(\alpha) G(\alpha)^{-1} S_n^+ G(\alpha) G(\alpha)^{-1} P_{0,M} G(\alpha) \\ &= e^{\alpha(\theta_m - \theta_n)}\ \mathrm{Tr}\ S_m^+ S_n^- P_{0,M}(\alpha), \end{aligned} \tag{3.5}$$

where we have written as

$$P_{0,M}(\alpha) := G(\alpha)^{-1} P_{0,M} G(\alpha). \tag{3.6}$$

This operator has the property,

$$\begin{aligned} P_{0,M}(\alpha)^2 &= G(\alpha)^{-1}P_{0,M}G(\alpha)G(\alpha)^{-1}P_{0,M}G(\alpha) \\ &= G(\alpha)^{-1}P_{0,M}^2G(\alpha) \\ &= P_{0,M}(\alpha). \end{aligned} \tag{3.7}$$

Using this, we have

$$\begin{aligned} \left|\mathrm{Tr}\ S_m^+S_n^-P_{0,M}(\alpha)\right| &= \left|\mathrm{Tr}\ S_m^+S_n^-P_{0,M}(\alpha)P_{0,M}(\alpha)\right| \\ &\le \sqrt{\mathrm{Tr}\ P_{0,M}(\alpha)^*S_n^+S_m^-S_m^+S_n^-P_{0,M}(\alpha)\cdot \mathrm{Tr}\ P_{0,M}(\alpha)^*P_{0,M}(\alpha)} \\ &\le \left\|S_m^+S_n^-\right\| \mathrm{Tr}\ P_{0,M}(\alpha)^*P_{0,M}(\alpha). \end{aligned} \tag{3.8}$$

From the definitions, (3.1) and (3.6), one has

$$\begin{aligned} \mathrm{Tr}\ P_{0,M}(\alpha)^*P_{0,M}(\alpha) &= \mathrm{Tr}\ G(\alpha)P_{0,M}G(\alpha)^{-1}G(\alpha)^{-1}P_{0,M}G(\alpha) \\ &= \mathrm{Tr}\ G(\alpha)^{-2}P_{0,M}G(\alpha)^2P_{0,M} \\ &\le q_M \left\|P_{0,M}(2\alpha)\right\|. \end{aligned} \tag{3.9}$$

Combining this, (3.3), (3.5) and (3.8), we have

$$\left|\left\langle S_m^+S_n^-\right\rangle_{0,M}\right| \le \left\|S_m^+S_n^-\right\| \left\|P_{0,M}(2\alpha)\right\| \exp\left[-\alpha\left(1-R^{-\kappa}\right)\{\mathrm{dist}(m,n)\}^{1-D/2}\right]. \tag{3.10}$$

Therefore it is sufficient to evaluate the norm $\|P_{0,M}(2\alpha)\|$ with a suitable choice of the parameter α.

Let us introduce the contour integral representation of the projection $P_{0,M}$ as

$$P_{0,M} = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-H_\Lambda}P_M, \tag{3.11}$$

where the closed path Γ is taken to encircle all of the eigenvalues in the sector of the corresponding ground state, and P_M is the projection onto the eigenspace $\mathcal{H}_{\Lambda,M}$ of S_Λ^z with the eigenvalue M. From the definition (3.6), one has

$$\begin{aligned} P_{0,M}(2\alpha) &= G(2\alpha)^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-H_\Lambda}P_MG(2\alpha) \\ &= G(2\alpha)^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-H_\Lambda}G(2\alpha)P_M \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-H'_\Lambda}P_M \end{aligned} \tag{3.12}$$

with the non-hermitian matrix $H'_\Lambda = G(2\alpha)^{-1}H_\Lambda G(2\alpha)$. In order to evaluate the resolvent $(z-H'_\Lambda)^{-1}$ in the right-hand side, we begin with getting the explicit form of the transformed Hamiltonian H'_Λ. Note that the Hamiltonian H_Λ^{XY} of (2.5) is written as

$$H_\Lambda^{\mathrm{XY}} = \sum_{\{i,j\}\in\Lambda_b} J_{i,j}^{\mathrm{XY}}(S_i^+S_j^- + S_i^-S_j^+). \tag{3.13}$$

Using the definition (3.3) of $\{\theta_i\}$ and the relations (3.4), one has

$$\begin{aligned} G(2\alpha)^{-1}H_\Lambda^{\mathrm{XY}}G(2\alpha) &= \sum_{\{i,j\}\in\Lambda_b} J_{i,j}^{\mathrm{XY}}\left[e^{2\alpha(\theta_i-\theta_j)}S_i^+S_j^- + e^{-2\alpha(\theta_i-\theta_j)}S_i^-S_j^+\right] \\ &= H_\Lambda^{\mathrm{XY}} + K_\Lambda + iL_\Lambda \end{aligned} \tag{3.14}$$

with the two hermitian matrices,

$$K_\Lambda = \sum_{\{i,j\}\in A_{1,R}(m)} J_{i,j}^{\mathrm{XY}}\left\{\cosh[2\alpha(\theta_i-\theta_j)]-1\right\}\left(S_i^+S_j^- + S_i^-S_j^+\right) \tag{3.15}$$

and

$$L_\Lambda = -i\sum_{\{i,j\}\in A_{1,R}(m)} J_{i,j}^{\mathrm{XY}}\sinh[2\alpha(\theta_i-\theta_j)]\left(S_i^+S_j^- - S_i^-S_j^+\right), \tag{3.16}$$

where the set $A_{1,R}(m)$ of the bonds is given by

$$A_{1,R}(m) = \{\{i,j\}\in\Lambda_b | 1\le \mathrm{dist}(i,m)\le R-1, \mathrm{dist}(j,m)\ge\mathrm{dist}(i,m)\}. \tag{3.17}$$

Thus we have

$$H'_\Lambda = H_\Lambda + K_\Lambda + iL_\Lambda. \tag{3.18}$$

The norms of the operators, K_Λ and L_Λ, can be estimated as follows:

Lemma 3.1 *The norm of the operator K_Λ satisfies*

$$\|K_\Lambda\| \le J_{\max}^{\mathrm{XY}}\left\|\left(S_i^+S_j^- + S_i^-S_j^+\right)\right\| C_0^2(\cosh 2\alpha - 1)\frac{2\kappa+D-1}{2\kappa+D-2}. \tag{3.19}$$

Remark: The bound implies that one can make the contribution from K_Λ small in the resolvent $(z-H'_\Lambda)^{-1}$ by choosing a small α.
Proof: From (3.15), one has

$$\|K_\Lambda\| \le J_{\max}^{\mathrm{XY}}\left\|\left(S_i^+S_j^- + S_i^-S_j^+\right)\right\| \times \sum_{\{i,j\}\in A_{1,R}(m)}\{\cosh[2\alpha(\theta_i-\theta_j)]-1\}. \tag{3.20}$$

The sum in the right-hand side is rewritten as

$$\sum_{\{i,j\}\in A_{1,R}(m)}\{\cosh[2\alpha(\theta_i-\theta_j)]-1\} = \sum_{r=1}^{R-1}\sum_{i:\mathrm{dist}(i,m)=r}\sum_{\substack{j:\{i,j\}\in\Lambda_b \\ \mathrm{dist}(j,m)=r+1}}\{\cosh[2\alpha(\theta_i-\theta_j)]-1\}. \tag{3.21}$$

From the definitions (3.2) and (3.3), one has

$$|\theta_i-\theta_j| \le \kappa R^{-\kappa+1-D/2}r^{\kappa-1} \le 1 \tag{3.22}$$

for i,j satisfying $\mathrm{dist}(i,m)=r$ and $\mathrm{dist}(j,m)=r+1$ for $r=1,2,\ldots,R-1$. From the second inequality,

$$\frac{\cosh[2\alpha(\theta_i-\theta_j)]-1}{4\alpha^2(\theta_i-\theta_j)^2} \le \frac{\cosh 2\alpha - 1}{4\alpha^2}. \tag{3.23}$$

Combining these inequalities, one obtains

$$\cosh[2\alpha(\theta_i - \theta_j)] - 1 \le (\cosh 2\alpha - 1)(\theta_i - \theta_j)^2 \le (\cosh 2\alpha - 1) R^{-2\kappa+2-D} r^{2\kappa-2}. \tag{3.24}$$

Substituting this into the right-hand side of (3.21), we have

$$\begin{aligned}
&\sum_{\{i,j\}\in A_{1,R}(m)} \{\cosh[2\alpha(\theta_i - \theta_j)] - 1\} \\
\le\ & (\cosh 2\alpha - 1) \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{j:\{i,j\}\in\Lambda_b \\ \mathrm{dist}(j,m)=r+1}} R^{-2\kappa+2-D} r^{2\kappa-2} \\
\le\ & C_0 (\cosh 2\alpha - 1) \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} R^{-2\kappa+2-D} r^{2\kappa-2} \\
\le\ & C_0^2 (\cosh 2\alpha - 1) \sum_{r=1}^{R-1} r^{D-1} R^{-2\kappa+2-D} r^{2\kappa-2} \\
\le\ & C_0^2 (\cosh 2\alpha - 1) \frac{2\kappa + D - 1}{2\kappa + D - 2},
\end{aligned} \tag{3.25}$$

where we have used the assumption (2.2) on the (fractal) dimension D and the definition (3.2) of the parameter κ. Substituting this into the right-hand side of (3.20) gives the desired bound (3.19). ■

In a similar way, we can obtain the following bound for $\|L_\Lambda\|$:

Lemma 3.2 *The norm of the operator L_Λ satisfies*

$$\|L_\Lambda\| \le J_{\max}^{\mathrm{XY}} \left\| \left(S_i^+ S_j^- - S_i^- S_j^+ \right) \right\| C_0^2 |\sinh 2\alpha| \left(1 + \frac{R^{D/2}}{\kappa + D - 1} \right). \tag{3.26}$$

Proof: In the same way as in the proof of Lemma 3.1, one has

$$\|L_\Lambda\| \le J_{\max}^{\mathrm{XY}} \left\| \left(S_i^+ S_j^- - S_i^- S_j^+ \right) \right\| \times \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{j:\{i,j\}\in\Lambda_b \\ \mathrm{dist}(j,m)=r+1}} |\sinh[2\alpha(\theta_i - \theta_j)]|. \tag{3.27}$$

For θ_i, θ_j in the sum, the following bound is valid:

$$\frac{|\sinh[2\alpha(\theta_i - \theta_j)]|}{|2\alpha(\theta_i - \theta_j)|} \le \frac{|\sinh 2\alpha|}{|2\alpha|}. \tag{3.28}$$

Therefore we have

$$|\sinh[2\alpha(\theta_i - \theta_j)]| \le |\sinh 2\alpha| |\theta_i - \theta_j| \le |\sinh 2\alpha| R^{-\kappa+1-D/2} r^{\kappa-1}. \tag{3.29}$$

Substituting this into the right-hand side of (3.27), we can obtain the desired bound (3.26) in the same way as in the proof of Lemma 3.1. ■

The upper bound (3.26) increases as the distance R between the two spins increases. In fact, this bound is not sufficient for estimating the resolvent $(z - H'_\Lambda)^{-1}$. We will further employ the technique developed in [32, 33]. For this purpose, we need the following estimate for the double commutator:

Lemma 3.3 *The following bound is valid:*

$$\|[L_\Lambda,[H_\Lambda,L_\Lambda]]\| \le C_1(\sinh 2\alpha)^2 \frac{2\kappa+D-1}{2\kappa+D-2}, \tag{3.30}$$

where C_1 is a positive constant which is independent of the parameter α.

Proof: Write the Hamiltonian H_Λ in terms of the local Hamiltonian $h_{u,v}$ as

$$H_\Lambda = \sum_{\{u,v\}\in\Lambda_b} h_{u,v} \quad \text{with} \quad h_{u,v} = J_{u,v}^{\mathrm{XY}}(S_u^+S_v^- + S_u^-S_v^+) + J_{u,v}^{\mathrm{Z}} S_u^z S_v^z. \tag{3.31}$$

Note that

$$\begin{aligned}
[H_\Lambda, L_\Lambda] &= -i \sum_{\{u,v\}\in\Lambda_b} \sum_{\{i,j\}\in A_{1,R}(m)} J_{i,j}^{\mathrm{XY}} \sinh[2\alpha(\theta_i-\theta_j)] \left[h_{u,v}, (S_i^+S_j^- - S_i^-S_j^+)\right] \\
&= -i \sum_{\{i,j\}\in A_{1,R}(m)} J_{i,j}^{\mathrm{XY}} \sinh[2\alpha(\theta_i-\theta_j)] \sum_{\{u,v\}\cap\{i,j\}\ne\emptyset} \left[h_{u,v}, (S_i^+S_j^- - S_i^-S_j^+)\right] \\
&= -i \sum_{\{i,j\}\in A_{1,R}(m)} J_{i,j}^{\mathrm{XY}} \sinh[2\alpha(\theta_i-\theta_j)] \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} M_{i,j;t},
\end{aligned} \tag{3.32}$$

where $M_{i,j;t}$ is a matrix with the support $\{i,j,t\}\subset\Lambda_s$. Using this, the double commutator is written as

$$\begin{aligned}
[L_\Lambda,[H_\Lambda,L_\Lambda]] &= -\sum_{\{k,\ell\}\in A_{1,R}(m)} \sum_{\{i,j\}\in A_{1,R}(m)} J_{k,\ell}^{\mathrm{XY}} J_{i,j}^{\mathrm{XY}} \sinh[2\alpha(\theta_k-\theta_\ell)] \sinh[2\alpha(\theta_i-\theta_j)] \\
&\quad \times \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \left[(S_k^+S_\ell^- - S_k^-S_\ell^+), M_{i,j;t}\right].
\end{aligned} \tag{3.33}$$

In the same way as in the proof of Lemma 3.1, we have

$$\begin{aligned}
\|[L_\Lambda,[H_\Lambda,L_\Lambda]]\| &\le \left(J_{\max}^{\mathrm{XY}}\right)^2 (\sinh 2\alpha)^2 R^{-2\kappa+2-D} \\
&\quad \times \sum_{r'=1}^{R-1} \sum_{k:\mathrm{dist}(k,m)=r'} \sum_{\substack{\{k,\ell\}\in\Lambda_b \\ \ell:\mathrm{dist}(\ell,m)=r'+1}} \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{\{i,j\}\in\Lambda_b \\ j:\mathrm{dist}(j,m)=r+1}} (r')^{\kappa-1} r^{\kappa-1} \\
&\quad \times \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \left\|\left[(S_k^+S_\ell^- - S_k^-S_\ell^+), M_{i,j;t}\right]\right\|.
\end{aligned} \tag{3.34}$$

We decompose the sum in the right-hand side into two parts, I_1 with $r > r'$ and I_2 with $r' \ge r$, as

$$\begin{aligned}
I_1 &= \sum_{r'=1}^{R-1} \sum_{k:\mathrm{dist}(k,m)=r'} \sum_{\substack{\{k,\ell\}\in\Lambda_b \\ \ell:\mathrm{dist}(\ell,m)=r'+1}} \sum_{r>r'}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{\{i,j\}\in\Lambda_b \\ j:\mathrm{dist}(j,m)=r+1}} (r')^{\kappa-1} r^{\kappa-1} \\
&\quad \times \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \left\|\left[(S_k^+S_\ell^- - S_k^-S_\ell^+), M_{i,j;t}\right]\right\|
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
I_2 &= \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{\{i,j\}\in\Lambda_b \\ j:\mathrm{dist}(j,m)=r+1}} \sum_{r'\ge r}^{R-1} \sum_{k:\mathrm{dist}(k,m)=r'} \sum_{\substack{\{k,\ell\}\in\Lambda_b \\ \ell:\mathrm{dist}(\ell,m)=r'+1}} (r')^{\kappa-1} r^{\kappa-1} \\
&\times \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \left\| \left[(S_k^+ S_\ell^- - S_k^- S_\ell^+), M_{i,j;t} \right] \right\|. \qquad (3.36)
\end{aligned}$$

First let us estimate I_2. Since $r' \ge r$ in the sum, one has $(r')^{\kappa-1} \le r^{\kappa-1}$. Using this inequality, I_2 is evaluated as

$$\begin{aligned}
I_2 \le\ & \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{\{i,j\}\in\Lambda_b \\ j:\mathrm{dist}(j,m)=r+1}} r^{2\kappa-2} \\
& \times \sum_{r'\ge r}^{R-1} \sum_{k:\mathrm{dist}(k,m)=r'} \sum_{\substack{\{k,\ell\}\in\Lambda_b \\ \ell:\mathrm{dist}(\ell,m)=r'+1}} \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \left\| \left[(S_k^+ S_\ell^- - S_k^- S_\ell^+), M_{i,j;t} \right] \right\| \\
\le\ & \sum_{r=1}^{R-1} \sum_{i:\mathrm{dist}(i,m)=r} \sum_{\substack{\{i,j\}\in\Lambda_b \\ j:\mathrm{dist}(j,m)=r+1}} r^{2\kappa-2} \\
& \times \sum_{t\in\Lambda_s:\mathrm{dist}(t,\{i,j\})=0,1} \sum_{\{k,\ell\}\cap\{i,j,t\}\ne\emptyset} \left\| \left[(S_k^+ S_\ell^- - S_k^- S_\ell^+), M_{i,j;t} \right] \right\| \\
\le\ & 12 C_0^4 \max\left\{ \left\| \left[(S_k^+ S_\ell^- - S_k^- S_\ell^+), M_{i,j;t} \right] \right\| \right\} \sum_{r=1}^{R-1} r^{D-1} r^{2\kappa-2} \\
\le\ & 12 C_0^4 \max\left\{ \left\| \left[(S_k^+ S_\ell^- - S_k^- S_\ell^+), M_{i,j;t} \right] \right\| \right\} \left[1 + \frac{R^{2\kappa+D-2}}{2\kappa+D-2} \right]. \qquad (3.37)
\end{aligned}$$

Similarly we have the same upper bound for I_1. Combining these bounds, (3.34), (3.35) and (3.36), the desired result (3.30) is obtained. ■

Now let us estimate $P_{0,M}(2\alpha)$ of (3.12). The contour integral in the right-hand side is written

$$\begin{aligned}
\int_\Gamma \frac{dz}{z - H'_\Lambda} P_M &= \int_{-y_0}^{y_0} \frac{i\,dy}{E_+ + iy - H'_\Lambda} P_M + \int_{E_+}^{E_-} \frac{dx}{x + iy_0 - H'_\Lambda} P_M \\
&+ \int_{y_0}^{-y_0} \frac{i\,dy}{E_- + iy - H'_\Lambda} P_M + \int_{E_-}^{E_+} \frac{dx}{x - iy_0 - H'_\Lambda} P_M. \qquad (3.38)
\end{aligned}$$

Here we choose the three real numbers, y_0, E_+ and E_-, as follows: Relying on Lemma 3.2, we take

$$y_0 = C_2 R^{D/2} \qquad (3.39)$$

satisfying

$$C_2 R^{D/2} - \|L_\Lambda\| \ge C_3 > 0 \qquad (3.40)$$

with some positive constants C_2 and C_3. From the assumption on the spectrum of the Hamiltonian H_Λ, we can take E_+, E_- so that the distance between the spectrum and $\{E_+, E_-\}$ is greater than or equal to $\Delta E/2$. From (3.38) and (3.39), we have

$$\begin{aligned}\|P_{0,M}(2\alpha)\| &\le \frac{C_2}{\pi}R^{D/2}\left\{\sup_{y\in[-y_0,y_0]}\|R'(E_+ + iy)P_M\| + \sup_{y\in[-y_0,y_0]}\|R'(E_- + iy)P_M\|\right\} \\ &+ \frac{E_+ - E_-}{2\pi}\left\{\sup_{x\in[E_-,E_+]}\|R'(x+iy_0)\| + \sup_{x\in[E_-,E_+]}\|R'(x-iy_0)\|\right\}, \qquad (3.41)\end{aligned}$$

where we have written $R'(z) = (H'_\Lambda - z)^{-1}$.

In order to estimate the norm of the resolvent $R'(z)$ with $z = E_+ + iy$, we employ the technique developed in [32, 33] for the following matrix element $\langle\Phi_+, L_\Lambda\Phi_-\rangle$:

Lemma 3.4 *Let $\Phi_- = P_{0,M}\Phi$ and $\Phi_+ = (1 - P_{0,M})\Phi$ for a vector $\Phi \in \mathcal{H}_{\Lambda,M}$. Then we have*

$$|\langle\Phi_+, L_\Lambda\Phi_-\rangle| \le f(\alpha,\Lambda)\|\Phi_+\|\|\Phi_-\|, \qquad (3.42)$$

where

$$f(\alpha,\Lambda) = \sqrt{\frac{1}{2\Delta E}\|[L_\Lambda,[H_\Lambda,L_\Lambda]]\| + 2\frac{\Delta\mathcal{E}}{\Delta E}\|L_\Lambda\|^2} \qquad (3.43)$$

with $\Delta\mathcal{E} = \max_{\mu,\mu'}\{|E_{0,\mu} - E_{0,\mu'}|\}$.

Proof: Note that

$$\begin{aligned}|\langle\Phi_+, L_\Lambda\Phi_-\rangle|^2 &= \langle\Phi_-, L_\Lambda\Phi_+\rangle\langle\Phi_+, L_\Lambda\Phi_-\rangle \\ &\le \|\Phi_+\|^2\langle\Phi_-, L_\Lambda(1-P_{0,M})L_\Lambda\Phi_-\rangle. \qquad (3.44)\end{aligned}$$

The matrix element in the right-hand side is evaluated as

$$\begin{aligned}\langle\Phi_-, L_\Lambda(1-P_{0,M})L_\Lambda\Phi_-\rangle &\le \frac{1}{\Delta E}\left\langle\Phi_-, L_\Lambda(H_\Lambda - \overline{E}_0)(1-P_{0,M})L_\Lambda\Phi_-\right\rangle \\ &= \frac{1}{\Delta E}\left\langle\Phi_-, L_\Lambda(H_\Lambda - \overline{E}_0)L_\Lambda\Phi_-\right\rangle \\ &- \frac{1}{\Delta E}\left\langle\Phi_-, L_\Lambda(H_\Lambda - \overline{E}_0)P_{0,M}L_\Lambda\Phi_-\right\rangle \\ &\le \frac{1}{\Delta E}\left\langle\Phi_-, L_\Lambda(H_\Lambda - \overline{E}_0)L_\Lambda\Phi_-\right\rangle + \frac{\Delta\mathcal{E}}{\Delta E}\|L_\Lambda\|^2\|\Phi_-\|^2, \end{aligned} \qquad (3.45)$$

where we have written $\overline{E}_0 = \sum_{\mu=1}^{q_M} E_{0,\mu}/q_M$. Further the first term in the right-hand side in the last line can be written

$$\left\langle\Phi_-, L_\Lambda(H_\Lambda - \overline{E}_0)L_\Lambda\Phi_-\right\rangle = \langle\Phi_-, L_\Lambda[H_\Lambda, L_\Lambda]\Phi_-\rangle + \left\langle\Phi_-, L_\Lambda^2(H_\Lambda - \overline{E}_0)\Phi_-\right\rangle. \qquad (3.46)$$

Therefore the matrix element can be evaluated as

$$\begin{aligned}
& \left|\left\langle \Phi_-, L_\Lambda (H_\Lambda - \overline{E}_0) L_\Lambda \Phi_- \right\rangle\right| \\
\le\ & \frac{1}{2}\left|\langle \Phi_-, L_\Lambda [H_\Lambda, L_\Lambda]\Phi_-\rangle + \langle L_\Lambda [H_\Lambda, L_\Lambda]\Phi_-, \Phi_-\rangle\right| + \Delta\mathcal{E}\|L_\Lambda\|^2\|\Phi_-\|^2 \\
=\ & \frac{1}{2}\left|\langle \Phi_-, [L_\Lambda [H_\Lambda, L_\Lambda]]\Phi_-\rangle\right| + \Delta\mathcal{E}\|L_\Lambda\|^2\|\Phi_-\|^2 \\
\le\ & \left(\frac{1}{2}\|[L_\Lambda [H_\Lambda, L_\Lambda]]\| + \Delta\mathcal{E}\|L_\Lambda\|^2\right)\|\Phi_-\|^2. \qquad (3.47)
\end{aligned}$$

Combining this, (3.44) and (3.45) gives the bound (3.42) with (3.43). ■

By using this lemma and the improved Combes-Thomas method [31], we obtain the following lemma:

Lemma 3.5 *Let $z = E_+ + iy$ with $y \in \mathbf{R}$. For a sufficiently large volume $|\Lambda_s|$, there exist positive constant, α_0 and C_4, such that*

$$\left\|(H'_\Lambda - z)^{-1} P_M\right\| \le C_4 \quad \text{for any} \ \ \alpha \le \alpha_0. \qquad (3.48)$$

Both of the constants, α_0 and C_4, are independent of the volume $|\Lambda_s|$ and of the distance R between the two spin operators, S_m^+, S_n^-.

Proof: Using the Schwarz inequality, one has

$$\|\Phi\|\,\|(H'_\Lambda - z)\Phi\| \ge \mathrm{Re}\,\langle(\Phi_+ - \Phi_-), (H'_\Lambda - z)(\Phi_+ + \Phi_-)\rangle \qquad (3.49)$$

for any vector $\Phi \in \mathcal{H}_{\Lambda,M}$, where $\Phi_+ = (1 - P_{0,M})\Phi$ and $\Phi_- = P_{0,M}\Phi$.

We recall the expression, $H'_\Lambda = H_\Lambda + K_\Lambda + iL_\Lambda$. For the hermitian part of $H'_\lambda - z$, one has

$$\begin{aligned}
& \mathrm{Re}\,\langle(\Phi_+ - \Phi_-), (H_\Lambda + K_\Lambda - E_+)(\Phi_+ + \Phi_-)\rangle \\
\ge\ & (E_1 - \|K_\Lambda\| - E_+)\|\Phi_+\|^2 + (E_+ - \|K_\Lambda\| - \max_\mu\{E_{0,\mu}\})\|\Phi_-\|^2 \\
+\ & \mathrm{Re}\left(\langle\Phi_+, K_\Lambda\Phi_-\rangle - \langle\Phi_-, K_\Lambda\Phi_+\rangle\right), \qquad (3.50)
\end{aligned}$$

where E_1 is the energy of the first excited state, and $E_{0,\mu}$ are the eigenvalues of the ground state. Since the last term in the right-hand side is equal to zero, one has

$$\mathrm{Re}\,\langle(\Phi_+ - \Phi_-), (H_\Lambda + K_\Lambda - E_+)(\Phi_+ + \Phi_-)\rangle \ge \left(\frac{1}{2}\Delta E - \|K_\Lambda\|\right)\|\Phi\|^2, \qquad (3.51)$$

where we have used the fact that the distance between the spectrum of $H_{\Lambda,M}$ and the energy E_+ is greater than or equal to $\Delta E/2$.

For the rest of $H'_\Lambda - z$, one has

$$\begin{aligned}
\mathrm{Re}\,\langle(\Phi_+ - \Phi_-), (iL_\Lambda - iy)(\Phi_+ + \Phi_-)\rangle &= -\mathrm{Im}\,\langle(\Phi_+ - \Phi_-), (L_\Lambda - y)(\Phi_+ + \Phi_-)\rangle \\
&= -\mathrm{Im}\left(\langle\Phi_+, L_\Lambda\Phi_-\rangle - \langle\Phi_-, L_\Lambda\Phi_+\rangle\right) \\
&= -2\ \mathrm{Im}\,\langle\Phi_+, L_\Lambda\Phi_-\rangle\,. \qquad (3.52)
\end{aligned}$$

From (3.42), (3.49), (3.51) and (3.52), one obtains

$$\begin{aligned} \|\Phi\|\|(H'_\Lambda - z)\Phi\| &\geq \left(\frac{1}{2}\Delta E - \|K_\Lambda\|\right)\|\Phi\|^2 - 2f(\alpha,\Lambda)\|\Phi_+\|\|\Phi_-\| \\ &= \left[\frac{1}{2}\Delta E - \|K_\Lambda\| - f(\alpha,\Lambda)\right]\|\Phi\|^2 + f(\alpha,\Lambda)\left(\|\Phi_+\| - \|\Phi_-\|\right)^2 \\ &\geq \left[\frac{1}{2}\Delta E - \|K_\Lambda\| - f(\alpha,\Lambda)\right]\|\Phi\|^2. \end{aligned} \tag{3.53}$$

From the assumption (2.3) on the quasidegeneracy of the ground state, Lemma 3.3 and the expression (3.43) of $f(\alpha,\Lambda)$, one can find that $f(\alpha,\Lambda)$ becomes small for a small parameter α, and for a sufficiently large volume $|\Lambda_s|$ compared to the distance R between the two spin operators. Combining this observation with Lemma 3.1, we have that there exist positive constants, α_0 and $\tilde{C}_4$, such that

$$\frac{1}{2}\Delta E - \|K_\Lambda\| - f(\alpha,\Lambda) \geq \tilde{C}_4 \tag{3.54}$$

for any $\alpha \leq \alpha_0$, and for a sufficiently large volume $|\Lambda_s|$ compared to the distance R between the two spin operators. Substituting this inequality into the right-hand side of the above bound (3.53), we obtain

$$\|(H'_\Lambda - z)\Phi\| \geq \tilde{C}_4\|\Phi\|. \tag{3.55}$$

Choosing $\Phi = (H'_\Lambda - z)^{-1}P_M\Psi$ with a vector Ψ, we have

$$\|P_M\Psi\| \geq \tilde{C}_4\left\|(H'_\Lambda - z)^{-1}P_M\Psi\right\|. \tag{3.56}$$

■

Similarly one can obtain the following two lemmas:

Lemma 3.6 *Let $z = E_- + iy$ with $y \in \mathbf{R}$. There exists a positive constant C_5 such that*

$$\left\|(H'_\Lambda - z)^{-1}P_M\right\| \leq C_5 \quad \text{for any} \quad \alpha \leq \alpha_0, \tag{3.57}$$

where α_0 is the same as in the preceding lemma.

Proof: Using the Schwarz inequality, one has

$$\begin{aligned} \|\Phi\|\,\|(H'_\Lambda - z)\Phi\| &\geq \mathrm{Re}\,\langle\Phi,(H'_\Lambda - z)\Phi\rangle \\ &\geq \left(\min_\mu\{E_{0,\mu}\} - \|K_\Lambda\| - E_-\right)\|\Phi\|^2 \\ &\geq \left(\frac{1}{2}\Delta E - \|K_\Lambda\|\right)\|\Phi\|^2 \end{aligned} \tag{3.58}$$

for any vector $\Phi \in \mathcal{H}_{\Lambda,M}$. Here we have used $\min_\mu\{E_{0,\mu}\} - E_- \geq \Delta E/2$. Therefore, in the same way as in the proof of the preceding lemma, one can prove the statement of the lemma. ■

Lemma 3.7 *Let* $z = x \pm iy_0$ *with* $x \in \mathbf{R}$. *Then*

$$\left\|(H'_\Lambda - z)^{-1}\right\| \le C_3^{-1}. \tag{3.59}$$

Proof: Using the Schwarz inequality and the definition (3.39) of y_0 with the condition (3.40), one has

$$\begin{aligned} \|\Phi\|\,\|(H'_\Lambda - z)\Phi\| &\ge |\mathrm{Im}\,\langle\Phi,(H'_\Lambda - z)\Phi\rangle| \\ &= |\langle\Phi,(L_\Lambda \mp y_0)\Phi\rangle| \\ &\ge |(y_0 - \|L_\Lambda\|)|\,\|\Phi\|^2 \ge C_3\|\Phi\|^2 \end{aligned} \tag{3.60}$$

for any vector Φ. Taking $\Phi = (H'_\Lambda - z)^{-1}\Psi$ with a vector Ψ, the desired bound can be obtained. ■

Combining the bound (3.41) with the three Lemmas 3.5, 3.6 and 3.7, we have

$$\|P_{0,M}(2\alpha)\| \le CR^{D/2} + C' \quad \text{for any} \quad \alpha \le \alpha_0 \tag{3.61}$$

with the positive constants C and C'. Substituting this into (3.10) and choosing $\alpha = \alpha_0$, we obtain the bound (2.10) for the spin-spin correlation.

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